A PARADOXICAL SOLUTION OF THE NAVIER-STOKES EQUATIONS

(ODNO PARADOKSAL'NOE RESHENIE URAVNENII NAV'E-STOKSA)

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We consider the interaction of an infinite plane with an infinite vortex filament passing through the origin of coordinates normal to the plane.

As a physical model of the problem under consideration we may consider a plate pierced by a slender rotating rod of great length. If the plane is absent, the fluid motion is determined by the laws

$$v_\varphi = \frac{C_0}{r}, \qquad P = P_\infty - \frac{\rho C_0^2}{2r^2}$$

where v_ϕ is the tangential component of the velocity vector, p_∞ the pressure at an infinite distance from the vortex filament. The vertical and radial components of velocity v_z and v_r will vanish.

Friction of the stream at the plane leads to secondary flow, the study of which is of interest in connection with the hydrodynamic processes in vortical or cyclonic combustion chambers. The problem may also be of interest for dynamic meteorology.

The origin of the secondary flow may be understood as follows. Fluid particles near the plane lose their circulatory velocity, and the centrifugal-force field consequently disappears at the plane. As a consequence of the predominant action of the pressure field near the plane, a flow arises in the direction of pressure drop, that is, toward the axis of the vortex, and with a velocity that is determined by the appearance of a shear force that compensates the loss in centrifugal force. As a result of continuity, fluid particles must acquire a motion along the axis away from the plane. An analogous picture of secondary flow arises also for a fluid rotating at infinity like a solid body [1].

1. Mathematical formulation of the problem and reduction of equations. Assuming the fluid incompressible and the motion steady and axisymmetric, we obtain the Navier-Stokes equations in a cylindrical coordinate system

$$
\begin{aligned}
&v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial r v_r}{\partial r} + \frac{\partial^2 v_r}{\partial z^2}\right) \\
&v_r \frac{\partial v_\varphi}{\partial r} + v_z \frac{\partial v_\varphi}{\partial z} + \frac{v_r v_\varphi}{r} = \nu\left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial r v_\varphi}{\partial r} + \frac{\partial^2 v_\varphi}{\partial z^2}\right), \qquad \frac{\partial r v_r}{\partial r} + \frac{\partial r v_z}{\partial z} = 0 \qquad (1.1) \\
&v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2}\right)
\end{aligned}
$$

The boundary conditions are

$$
\begin{aligned}
v_r = v_\varphi = v_z = 0 \qquad & \text{at } z = 0 \\
v_\varphi = \frac{C_0}{r}, \qquad P = P_\infty - \frac{\rho C_0^2}{2r^2} \qquad & \text{at } z = \infty
\end{aligned}
$$

the fluid is at rest at infinity, that is

$$
v_r = v_\varphi = v_z = 0, \qquad P = P_\infty \qquad \text{at } r = \infty
$$

and at $r = 0$ the component v_z is finite and $v_r = 0$ (the condition that neither sources nor sinks exist).

We introduce the dimensionless functions

$$
u = \frac{r v_r}{C_0}, \qquad \Phi = \frac{r v_\varphi}{C_0}, \qquad w = \frac{r v_z}{C_0}, \qquad \pi = \frac{r^2 (P - P_\infty)}{\rho C_0^2}
$$

Then the system (1.1) transforms to

$$
\begin{aligned}
&u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{u^2 + \Phi^2}{r} = -\frac{\partial \pi}{\partial r} + \frac{2\pi}{r} + \frac{\nu}{C_0} r\left(\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}\right) \\
&u\frac{\partial \Phi}{\partial r} + w\frac{\partial \Phi}{\partial z} = \frac{\nu}{C_0} r\left(\frac{\partial^2 \Phi}{\partial r^2} - \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial z^2}\right), \qquad \frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} = 0 \qquad (1.2) \\
&u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} - \frac{uw}{r} = -\frac{\partial \pi}{\partial z} + \frac{\nu}{C_0} r\left(\frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\frac{\partial w}{\partial r} + \frac{w}{r^2} + \frac{\partial^2 w}{\partial z^2}\right)
\end{aligned}
$$

The corresponding boundary conditions take the form

$$
\begin{aligned}
u = \Phi = w = 0 \qquad & \text{at } z = 0 \\
\Phi = 1, \qquad \pi = -\frac{1}{2} \qquad & \text{at } z = \infty \\
u = w = 0 \qquad & \text{at } r = 0
\end{aligned}
$$

We seek a solution of the system (1.2) having the following properties:

(a) The functions u, Φ, w, π should be continuous in the entire closed half-space except, perhaps, the origin of coordinates and the "point" with coordinates $r = \infty$, $z = \infty$.

(b) As the boundaries of the half-space are approached, these functions should tend to the limits determined by the boundary conditions.

The dimensionless dependent variables u, Φ, w, π should be functions of all possible dimensionless combinations formed from the quantities r, z, ν and C_0. From these four quantities it is possible to form two and only two independent dimensionless combinations

$$\frac{\nu}{C_0} = k, \qquad \frac{z}{r} = \eta$$

The quantity $1/k$ plays the role of a Reynolds number. It can therefore be asserted that any of the functions u, Φ, w and π cannot depend on r and z separately, but only on the single combination η. The number k plays the role of a parameter. Thus the problem posed has been reduced to the class of self-similar problems [2].

We introduce η into the system (1.2), bearing in mind that

$$\frac{\partial}{\partial r} = -\frac{z}{r^2}\frac{d}{d\eta}, \qquad \frac{\partial}{\partial z} = \frac{1}{r}\frac{d}{d\eta}, \qquad \frac{\partial^2}{\partial r^2} = \frac{z^2}{r^4}\frac{d^2}{d\eta^2} + \frac{2z}{r^3}\frac{d}{d\eta}, \qquad \frac{\partial^2}{\partial z^2} = \frac{1}{r^2}\frac{d^2}{d\eta^2}$$

Then we obtain the equations

$$u'(w - \eta u) - u^2 - \Phi^2 = \eta\pi' + 2\pi + k\,[(1+\eta^2)u'' + 3\eta u'] \qquad (1.3)$$

$$\Phi'(w - \eta u) = k\,[(1+\eta^2)\,\Phi'' + 3\eta\Phi'] \qquad (1.4)$$

$$w'(w - \eta u) = -\pi' + k\,[(1+\eta^2)\,w'' + 3\eta w' + w] \qquad (1.5)$$

$$w' = \eta u' \qquad (1.6)$$

In transforming the boundary conditions we note that by the introduction of the single variable η the points (∞, z) and $(r, 0)$ coalesce into the single point $\eta = 0$, and the points $(0, z)$ and (r, ∞) coalesce into the point $\eta = \infty$. (Here the argument η is indeterminate at the points $(0, 0)$ and (∞, ∞).) Consequently

$$u = \Phi = w = 0 \qquad \text{at } \eta = 0 \qquad (1.7)$$

$$u = w = 0, \qquad \Phi = 1, \qquad \pi = -\frac{1}{2} \qquad \text{at } \eta = \infty \qquad (1.8)$$

We observe a certain relation between the functions u and w that is indispensable for what follows. It can be shown that

$$\lim_{\eta\to\infty}(w - \eta u) = \lim_{\substack{r\to 0\\ z\neq 0}} \frac{rv_z - zv_r}{C_0} = 0$$

so that

$$w - \eta u = 0 \qquad \text{at } \eta = \infty \tag{1.9}$$

Because according to the condition (1.7), $u = 0$ at $r = \infty$ for all $z \neq \infty$, it follows that

$$\lim_{\eta\to 0}\frac{dw}{d\eta} = \lim_{\eta\to 0}\eta u' = \lim_{\substack{r\to\infty \\ z\neq\infty}} z\frac{\partial u}{\partial z} = 0, \quad \text{at} \quad w'(0) = 0 \tag{1.10}$$

We obtain from (1.5) the indefinite integral with respect to η

$$\pi = -w(w - \eta u) + k\,[(1 + \eta^2)w' + \eta w] + C_1 \tag{1.11}$$

For the determination of C_1 we take $\eta \to \infty$. Then in view of (1.7) and (1.9) the first term of (1.11) vanishes; we calculate

$$M = \lim_{\eta\to\infty}(\eta^2 w' + \eta w) = \lim_{\eta\to\infty}\eta(\eta w)' = \lim_{\eta\to\infty}(\pi - C_1) = -\tfrac{1}{2} - C_1$$

so that the limit sought exists. We consider the limit

$$L = \lim_{\eta\to\infty}\frac{\eta w}{\ln\eta}$$

for the calculation of which it is possible to apply L'Hopital's rule*; since the limit of the ratio of derivatives

$$\lim_{\eta\to\infty}\frac{(\eta w)'}{(\ln\eta)'} = \lim_{\eta\to\infty}\eta(\eta w)' = M$$

exists under these conditions, it follows that $L = M$. On the other hand

$$L = \lim_{r\to 0}\frac{zv_z}{C_0}\Big/\lim_{\eta\to\infty}\ln\eta = 0 \tag{1.12}$$

since v_z is bounded at $r = 0$. Thus $M = 0$ and $C_1 = -1/2$.

We introduce a new variable and a new function

$$x = \frac{\eta}{\sqrt{1+\eta^2}}, \qquad y = \frac{1}{\sqrt{1+\eta^2}}(w - \eta u) = \sqrt{1-x^2}\left(w - \frac{x}{\sqrt{1-x^2}}u\right) \tag{1.13}$$

Then proceeding from (1.6) it is not difficult to obtain

$$u = -(1 - x^2)\,y' - xy, \qquad w = \sqrt{1-x^2}\,(y - xy') \tag{1.14}$$

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* For the conditions of applicability of L'Hopital's rule see, for example, [3]. Attention should be drawn to the comment on page 321.

where primes denote differentiation with respect to x.

If (1.11) is substituted into (1.3), Equations (1.3) and (1.4) can be transformed into the form

$$-k\,(1-x^2)^2\,y''' = 1-\Phi^2-\frac{1-x^2}{2}(y^2)''-x\,(y^2)'+y^2 \qquad (1.15)$$

$$y\Phi' = k\,(1-x^2)\,\Phi'' \qquad (1.16)$$

The boundary conditions are

$$y\,(0)=0, \qquad y\,(1)=0, \qquad y'\,(0)=0 \qquad (1.17)$$

$$\Phi\,(0)=0, \qquad \Phi\,(1)=1 \qquad (1.18)$$

The first of conditions (1.17) follows from (1.13) and (1.7), the second from (1.13) and (1.9), and the third from the first equation (1.14) together with (1.7); the conditions (1.18) follow from (1.7) and (1.8).

These conditions are just sufficient, because the system (1.15)-(1.16) is equivalent to one fifth-order equation.

Differentiating Equation (1.15) gives

$$-k\,(1-x^2)^2\,y^{\mathrm{IV}}+4kx\,(1-x^2)\,y''' = -2\,\Phi\Phi'-\frac{1-x^2}{2}(y^2)''' \qquad (1.19)$$

or

$$-k\,(1-x^2)\,y^{\mathrm{IV}}+4kxy''' = -2\frac{\Phi\Phi'}{1-x^2}-\frac{1}{2}\,(y^2)''' \qquad (1.20)$$

We integrate (1.19) by parts

$$-k\,(1-x^2)\,y'''+2kxy''-2ky' = -2\int_0^x\frac{\Phi\Phi'}{1-x^2}dx-\frac{1}{2}(y^2)''+C_2 \quad (1.20)$$

We then integrate (1.20) twice

$$-k\,(1-x^2)\,y'-2kxy = -2\int_0^x dx\int_0^x dx\int_0^x\frac{\Phi\Phi'}{1-x^2}dx-\frac{y^2}{2}+\frac{C_2x^2}{2}-\frac{C}{2}x+C_3 \qquad (1.21)$$

Now setting $x=0$ we find, by virtue of the conditions $y(0)=0$ and $y'(0)=0$, that $C_3=0$.

The constant C_2 is "superfluous", since it originates from the fact that the order of Equation (1.15) was increased by differentiation. To determine C_2 we compare (1.15) and (1.20) at $x=0$. In so doing it is necessary to take into account the fact that $xy'' = 0$ at $x = 0$, which is easily seen by differentiating the second equation (1.14) and using (1.10).

Setting $x = 0$ in (1.15) and (1.20), we find

$$-ky''' (0) = 1 - \tfrac{1}{2} y^{2\prime\prime} (0), \qquad -ky''' (0) = -\tfrac{1}{2} y^{2\prime\prime} (0) + C_2$$

Comparing these expressions we obtain $C_2 = 1$. Consequently, (1.21) can be written in the form

$$2k(1-x^2)y' + 4kxy - y^2 = 4\int_0^x dx \int_0^x dx \int_0^x \frac{\Phi\Phi'}{1-x^2} dx - x^2 + Cx = F(x) \quad (1.22)$$

To determine the constant C we use the second condition (1.17), which gives $F(1) = 0$. Thus reverting to the variable η and Equation (1.13) we find

$$(1-x^2)\frac{dy}{dx} = -u - \frac{\eta}{1+\eta^2}(w - \eta u)$$

Hence in virtue of (1.8) and (1.9) we have that $(1 - x^2)y' \to 0$ as $\eta \to \infty$ $(x \to 1)$.

Integrating (1.16) we obtain in succession

$$\Phi' = a \exp \int_0^x \frac{y\, dx}{k(1-x^2)}, \qquad \Phi = a \int_0^x \left[\exp \int_0^x \frac{y\, dx}{k(1-x^2)}\right] dx \quad (1.23)$$

The constant a should be determined from the last of conditions (1.18), We note that $a = \Phi'(0)$. Further, let

$$y = 2k(1-x^2) S \quad (1.24)$$

Then from (1.22) and (1.23) we have

$$S' = S^2 + \frac{F(x)}{4k^2(1-x^2)^2}, \qquad \Phi = a\int_0^x \left[\exp \int_0^x 2S\, dx\right] dx \quad (1.25)$$

Here

$$a = \left\{\int_0^1 \left[\exp \int_0^x 2S\, dx\right] dx\right\}^{-1}, \qquad S(0) = 0 \quad (1.26)$$

where the last condition follows from (1.17).

2. Analysis of the equations. In what follows particular significance is attached to the sign of the function

$$F(x)=4\int_0^x dx\int_0^x dx\int_0^x \frac{\Phi\Phi'}{1-x^2}dx-x^2+Cx \tag{2.1}$$

The function $y(x)$ is continuous in the entire closed interval (0, 1). Therefore according to (1.23) $\Phi'(x)$ is continuous in the interval $0 \leqslant x < 1$. Also, Φ' cannot change sign. Consequently the function $\Phi(x)$ is monotonic. But since $\Phi(0) = 0$ and $\Phi(1) = 1$,

$$\Phi(x)\leqslant 1 \quad \text{for} \quad 0\leqslant x\leqslant 1 \tag{2.2}$$

We transform (2.1), using the formula for transformation of a multiple integral into a single integral, giving

$$F(x)=2\int_0^x \frac{(x-t)^2}{1-t^2}\Phi\Phi'\,dt-x^2+Cx$$

or

$$F(x)=2\int_0^x \frac{(x-t)(1-tx)}{(1-t^2)^2}\Phi^2dt-x^2+Cx \tag{2.3}$$

Determining C from the condition $F(1) = 0$, we find

$$F(x)=x-x^2-2(1-x)^2\int_0^x \frac{t\Phi^2}{(1-t^2)^2}dt-2x\int_x^1 \frac{\Phi^2dt}{(1+t)^2} \tag{2.4}$$

The last two terms in (2.4) are strictly negative; hence if the function Φ is replaced in (2.4) by its upper limit $\Phi = 1$, the right-hand side of (2.4) is not thereby increased, that is

$$F(x)\geqslant x-x^2-2(1-x)^2\int_0^x \frac{t\,dt}{(1-t^2)^2}-2x\int_x^1 \frac{dt}{(1+t)^2}\equiv 0 \tag{2.5}$$

Thus $F(x) > 0$ for $0 < x < 1$; considering (2.5) it follows from (1.25), on the basis of Chaplygin's theorem on differential inequalities [4], that $S > 0$, and consequently also $y > 0$.

Since $\Phi' > 0$, it follows from (1.16) that also $\Phi'' > 0$.

The results obtained permit establishing the inequality

$$\Phi(x)\leqslant x \quad \text{for} \quad 0\leqslant x\leqslant 1 \tag{2.6}$$

In fact, (2.6) is equivalent to the inequality

$$H(x) = \Phi(x) - x \leqslant 0$$

The function $H(x)$ has a continuous derivative and vanishes at the points $x = 0$ and $x = 1$; consequently, according to Rolle's theorem, it has at least one extremum in the interval $0 < x < 1$. The condition $\Phi'' > 0$ establishes that there is only one such extremum, because between two extrema there must be an inflection point, which is excluded by the condition $\Phi'' > 0$. Furthermore the condition $\Phi'' > 0$ also shows that $H(x)$ has a minimum. But a function having a single minimum in the interval [0, 1] and vanishing at the end points is necessarily negative, which also proves the inequality (2.6).

From the inequality (2.6) follows in an obvious fashion the first paradoxical result

$$a \leqslant 1 \quad \text{for arbitrary } k \tag{2.7}$$

which contradicts the ideas of boundary-layer theory, according to which $a \sim 1/\sqrt{k}$ so that as $k \to 0$ the quantity a should increase without limit (see [5], where a problem analogous to that considered here is solved by Pohlhausen's method).

The inequality (2.6) permits the inequality (2.5) to be made more precise. Replacing the function $\Phi(t)$ by t in (2.4), we obtain the inequality

$$F(x) \geqslant (4 \ln 2 - 2)\, x + 2x^2 - (1 - x)^2 \ln (1 - x) - (1 + x)^2 \ln (1 + x) = F_1(x) \tag{2.8}$$

The function (2.8) is inconvenient for what follows; we therefore introduce

$$F_2(x) = \frac{1}{2} x (1 - x^2)^2 \tag{2.9}$$

It can be shown that $F_1(x) > F_2(x)$ in the interval $0 < x < 1$ (this follows graphically from Fig. 1). But, as is easily seen

$$\lim_{x \to 1} \frac{F_2(x)}{F_1(x)} = 0$$

Therefore the inequality $F_2(x) < F_1(x)$ is satisfied everywhere on the interval [0, 1], or taking (2.8) into account

$$F(x) \geqslant F_2(x) \quad \text{for } 0 \leqslant x \leqslant 1 \tag{2.10}$$

We consider the equation

$$\tau' = \tau^2 + \frac{F_2(x)}{4k^2(1-x^2)^2} \tag{2.11}$$

The solution of (2.11) satisfying the condition $\tau(0) = 0$ has the form

$$\tau = \frac{3}{2}\varkappa\sqrt{x}\,\frac{J_{2/3}(\varkappa x^{3/2})}{J_{-1/3}(\varkappa x^{3/2})} \qquad \left(\varkappa = \frac{1}{3k\sqrt{2}}\right) \tag{2.12}$$

Comparing (2.11) with (1.25), in virtue of the inequality (2.10) we conclude on the basis of Chaplygin's theorem on differential inequalities that

$$S(x) \geqslant \tau(x) \quad \text{for} \quad 0 \leqslant x < 1 \tag{2.13}$$

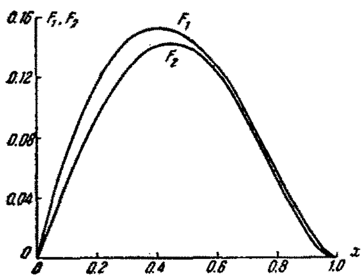

Fig. 1.

But the inequality (2.13) cannot be satisfied for arbitrary values of the parameter k. Indeed, (2.12) represents a meromorphic function having poles at the points

$$x_n = (3k\sqrt{2}\mu_n)^{2/3} \tag{2.14}$$

where μ_n are the roots of the equation $J_{-1/3}(\mu) = 0$. Since $S(x)$ is a continuous function in the interval [0, 1] it is necessary, in order for the inequality to be satisfied, to require in any case that the first pole of the function (2.12) lie outside the interval [0, 1], that is

$$x_1 > 1, \quad \text{or} \quad k > \frac{1}{3\sqrt{2}\mu_1} \approx \frac{1}{8} \tag{2.15}$$

If the condition (2.15) is not satisfied, $S(x)$ cannot be a continuous function in the whole interval [0,1]. Thus a second paradoxical result is found: for Reynolds numbers greater than 8 the problem posed does not have a bounded solution.

3. Proof of existence and uniqueness of solution for small Reynolds number. We prove that if $1/k \leqslant 4.8096$ then: (a) under the condition $y(0) = 0$ the system of equations (1.22) and (1.23) has a solution that is unique and continuous in the interval [0,1]; (b) as $x \to 1$ the function $y(x)$ has a limit equal to zero; (c) all the initial conditions for the functions v_r, v_z, v_ϕ and p are satisfied. We will solve the system of equations (1.25) and (2.4) by the method of successive approximations based on the following scheme:

$$\Phi_n = \left(\int_0^x \left[\exp\int_0^x 2S_{n-1}dx\right]dx\right) \Big/ \left(\int_0^1 \left[\exp\int_0^x 2S_{n-1}dx\right]dx\right) \tag{3.1}$$

$$F_n = x - x^2 - 2(1-x)^2 \int_0^x \frac{t\Phi_n^2}{(1-t^2)^2}\, dt - 2x \int_x^1 \frac{\Phi_n^2}{(1+t)^2}\, dt \tag{3.2}$$

$$S_0 = 0, \qquad S_n = \int_0^x \left[S_{n-1}^2 + \frac{F_n(x)}{4k^2(1-x^2)^2} \right] dx \tag{3.3}$$

$$\Phi_1 = x$$

We have

$$F_1 = (4 \ln 2 - 2)\, x + 2x^2 - (1-x)^2 \ln(1-x) - (1+x)^2 \ln(1+x)$$

$$S_1 = \frac{1}{4k^2}\left[\frac{x}{1+x} + \frac{1}{2}\ln(1+x) - \frac{1+2x}{1+x}\ln 2 - \frac{1}{1-x}\ln\frac{1+x}{2} + \right.$$
$$\left. + \frac{1}{2}\,\frac{1-x}{1+x}\ln(1-x)\right]$$

As is evident, $S_1(x)$ is a continuous function in the entire interval $[0,1]$, with $S_1(1) = (1 - \ln 2)/4k^2$. Thus all functions in the first approximation are continuous for $0 \leqslant x \leqslant 1$. Let us assume that the functions S_{n-1}, F_{n-1}, Φ_{n-1} also possess this same property. We prove that S_n, F_n, Φ_n are then also continuous in the interval $[0,1]$.

The continuity of $\Phi_n(x)$ and $\Phi_n'(x)$ follows immediately from (3.1), with $\Phi_n(x) \leqslant 1$ and $\Phi_n'(x) \leqslant M_n$, where M_n is the maximum value of the positive function $\Phi_n(x)$.

From (3.2) we have $F_n(1) = 0$. We compute $F_n'(1)$.

$$F_n'(1) = -1 + 4\lim_{x\to 1}\left[(1-x)\int_0^x \frac{t\Phi_n^2 dt}{(1-t^2)^2}\right] = -1 + \Phi_n^2(1) = 0$$

because $\Phi_n(1) = 1$. Using (2.1) we find

$$F_n'' = 4\int_0^x \frac{\Phi_n \Phi_n'}{1-x^2}\, dx - 2$$

Substituting here the estimates obtained for Φ_n and Φ_n' we obtain

$$F_n'' \leqslant 2M_n \ln\frac{1+x}{1-x} - 2$$

Integrating this inequality twice between the limits x and 1, we find

$$F_n(x) \leqslant 2M_n\left\{\frac{(1-x)^2}{4}[1 - 2\ln(1-x)] + 1 - 2x\ln 2 + \right.$$

$$+\frac{(1+x)^2}{4}[2\ln(1+x)-1]\Big\}-(1-x)^2=2M_n\Big\{\frac{(1-x)^2}{4}[1-2\ln(1-x)]+$$
$$+\frac{1+\ln 2}{2}(1-x^2)-\frac{1}{12}(1-x^3)-\ldots\Big\}-(1-x)^2 \qquad (3.4)$$

Hence

$$F_n(x)\leqslant M_n(1-x^2)^2\left[\frac{3+2\ln 2}{2}-\ln(1-x)\right] \qquad (3.4)$$

In virtue of the fact that $\Phi_n(x) \leqslant 1$, it follows from (3.2) that $F_n(x) \geqslant 0$.

Substituting the estimates obtained for $F_n(x)$ into (3.3), we find that $S_n(x)$ is continuous everywhere in the interval [0,1], since according to (3.4) the expression $F_n(x)/[4k^2(1-x^2)^2]$ has an integrable singularity at the point $x = 1$. Hence by the method of induction we have proved the continuity of all approximations in the interval [0,1]. We prove their convergence under the condition $1/k < 4.8096$.

In virtues of the facts that $S_0 = 0$ and $F_1(x) \geqslant 0$ we conclude from (3.3) that

$$S_1 \geqslant S_0 = 0 \qquad (3.5)$$

We show that if $S_n > S_{n-1}$ then $\Phi_{n+1} < \Phi_n$. We consider the difference

$$\Phi_{n+1}-\Phi_n=\frac{\Psi_{n+1}(x)}{\Psi_{n+1}(1)}-\frac{\Psi_n(x)}{\Psi_n(1)}=\frac{\Psi_{n+1}(x)\Psi_n(1)-\Psi_{n+1}(1)\Psi_n(x)}{\Psi_{n+1}(1)\Psi_n(1)}$$

Here and henceforth we introduce the notation

$$\Psi_n(x)=\int_0^x\psi_{n-1}(x)\,dx,\quad \psi_n(x)=\exp\int_0^x 2S_n dx$$

In virtue of the inequality $S_n > S_{n-1}$ we can write

$$S_n = S_{n-1}+\delta(x)$$

where $\delta(x) \geqslant 0$. In this case

$$\psi_n=\exp\int_0^x 2\delta dx\exp\int_0^x 2S_{n-1}dx=\varphi(x)\psi_{n-1}$$

where the functions $\phi(x)$ and ψ_{n-1} are continuous and positive, and moreover the function $\phi(x)$ is non-decreasing, since

$$\varphi'(x) = 2\delta \exp \int_0^x 2\delta dx \geqslant 0$$

On the basis of the theorem of the mean we have

$$\int_0^x \varphi(t) \psi_{n-1} dt = \varphi(\theta) \int_0^x \psi_{n-1} dx \qquad (0 \leqslant \theta \leqslant x) \tag{3.6}$$

Considering this we find

$$\Phi_{n+1} - \Phi_n = \frac{\varphi(\theta_1) - \varphi(\theta_2)}{\varphi(\theta_2)} \frac{\psi_n(x)}{\psi_n(1)} \quad \begin{pmatrix} 0 \leqslant \theta_1 \leqslant x \\ 0 \leqslant \theta_2 \leqslant 1 \end{pmatrix} \tag{3.7}$$

We prove that $\theta_1 < \theta_2$. Assuming that $\theta = \theta(x)$, we differentiate (3.6) with respect to x. We have

$$\varphi(x) \psi_{n-1}(x) = \varphi(\theta) \psi_{n-1}(x) + \frac{d\varphi}{d\theta} \frac{d\theta}{dx} \int_0^x \psi_{n-1} dx$$

In view of the fact that $\phi(x) \geqslant \phi(\theta)$ and $d\phi / d\theta \geqslant 0$, we conclude that $d\theta / dx \geqslant 0$; since $x < 1$, this proves that $\theta_1 < \theta_2$, from which $\phi(\theta_1) < \phi(\theta_2)$, and consequently also, according to (3.7),

$$\Phi_{n+1} \leqslant \Phi_n \tag{3.8}$$

Using the estimate (3.8) we obtain, according to (3.2),

$$F_{n+1} \geqslant F_n \tag{3.9}$$

Proceeding from (3.5) we find successively $\Phi_2 < \Phi_1$, $F_2 > F_1$, and according to (3.3) $S_2 > S_1$. We assume that $S_n > S_{n-1}$; then (3.8) and (3.9) are valid. Using (3.3) we form the difference

$$S_{n+1} - S_n = \int_0^x \left[(S_n^2 - S_{n-1}^2) + \frac{F_{n+1} - F_n}{4k^2 (1 - x^2)^2} \right] dx \geqslant 0, \quad S_{n+1} \geqslant S_n$$

Thus it is proved by the method of induction that all the successive approximations form the sequences

$$\Phi_1 \geqslant \Phi_2 \geqslant \ldots \geqslant \Phi_n \geqslant \ldots \tag{3.10}$$

$$0 \leqslant F_1 \leqslant F_2 \leqslant \ldots \leqslant F_n \leqslant \ldots \tag{3.11}$$

$$0 \leqslant S_1 \leqslant S_2 \leqslant \ldots \leqslant S_n \leqslant \ldots \tag{3.12}$$

But according to (3.1) the function $\Phi_n > 0$. Therefore the sequence of functions (3.10) has a limit

$$\Phi(x) = \lim \Phi_n(x) \qquad \text{as } n \to \infty \tag{3.13}$$

Again by virtue of the inequality $\Phi_n \geqslant 0$ we conclude from (3.2) that

$$F_n \leqslant x - x^2 \leqslant \frac{1+x}{4}(1-x^2) \tag{3.14}$$

The bound (3.14) means that the sequence (3.11) also has a limit

$$F(x) = \lim F_n(x), \quad \text{as } n \to \infty \tag{3.15}$$

Substituting (3.14) into (3.3) we obtain

$$S_n \leqslant \int_0^x \left[S_{n-1}^2 + \frac{1}{16k^2(1-x)}\right] dx \tag{3.16}$$

We consider the equation

$$\sigma' = \sigma^2 + \frac{1}{16k^2(1-x)}, \qquad \sigma(0) = 0 \tag{3.17}$$

Solving (3.17) by the method of successive approximations, we obtain

$$\sigma_0 = 0, \qquad \sigma_n = \int_0^x \left[\sigma_{n-1}^2 + \frac{1}{16k^2(1-x)}\right] dx \tag{3.18}$$

We subtract from the inequality (3.16) the equality (3.18)

$$S_n - \sigma_n \leqslant \int_0^x (S^2{}_{n-1} - \sigma^2{}_{n-1})\, dx \tag{3.19}$$

We have

$$S_1 - \sigma_1 \leqslant 0, \qquad S_2 - \sigma_2 \leqslant \int_0^x (S_1{}^2 - \sigma_1{}^2)\, dx \leqslant 0 \text{ etc.}$$

It is easy to see that in general $S_n \leqslant \sigma_n$, and also

$$0 \leqslant \sigma_1 \leqslant \sigma_2 \leqslant \ldots \leqslant \sigma_n \leqslant \ldots$$

But according to Picard's theorem σ_n converges in a certain interval I to the exact solution σ, which is easily found by solving (3.17):

$$\sigma = \frac{b^2}{2t} \frac{J_0(t) - Y_0(t)\, J_0(b) / Y_0(b)}{J_1(t) - Y_1(t)\, J_0(b) / Y_0(b)} \qquad \left(t = \frac{\sqrt{1-x}}{2k}, \; b = \frac{1}{2k}\right) \tag{3.20}$$

Noting that as x varies in the interval $[0,1]$ the variable t varies between the limits $0 \leqslant t \leqslant b$, we find the roots of the denominator of (3.20), that is, we solve the equation

$$\frac{J_1(t)}{Y_1(t)} = \frac{J_0(b)}{Y_0(b)} \tag{3.21}$$

We solve (3.21) graphically. Figure 2 shows graphs of the functions

$J_0(t)/Y_0(t)$ and $J_1(t)/Y_1(t)$. To determine the first root of Equation (3.21) we proceed as follows. We take the point $t = b$ and find the value $J_0(b)/Y_0(b)$. Then we draw a horizontal line corresponding to this value to its intersection with the curve $J_1(t)/Y_1(t)$. The abscissa of the point of intersection gives the desired root t_1. From examination of Fig. 2 it is not difficult to convince oneself that if $b < \lambda_1$, where λ_1 is the first root of the equation $J_0(\lambda) = 0$, then the root t_1, lies outside the interval $[0, b]$. Consequently, for

$$b = \frac{1}{2k} < \lambda_1 = 2.4048 \tag{3.22}$$

Equation (3.31) has no root in the interval $[0,1]$.

This shows that under the condition (3.22) the function (3.20) has no singularities in the open interval $0 \leqslant x < 1$. We investigate the behavior of this function in the neighborhood of the point $x = 1$ $(t = 0)$. Using for the functions $Y_0(t)$ and $Y_1(t)$ the representations

$$Y_0(t) \sim \frac{2}{\pi} \ln \frac{t}{2}, \qquad Y_1(t) \sim -\frac{2}{\pi}\frac{1}{t} \qquad \text{as } t \to 0$$

we find that as $t \to 0$

$$\sigma \sim -\frac{b^2}{2} \ln \frac{t}{2} = -\frac{1}{8k^2} \ln \frac{\sqrt{1-x}}{2k} \tag{3.23}$$

Thus under condition (3.22) the function $\sigma(x)$ is continuous in the interval $[0,1]$ and has a logarithmic singularity at the point $x = 1$. This result allows the assertion that the interval of convergence I of the successive approximations σ_n can be extended up to the point $1 - \epsilon$, where $0 < \epsilon < 1$. Considering this we find

$$0 \leqslant S_n \leqslant \sigma_n \leqslant \sigma \quad \text{for } 0 \leqslant x \leqslant 1 - \varepsilon \tag{3.24}$$

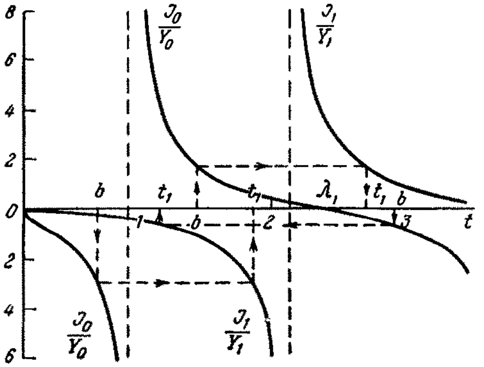

Fig. 2.

The non-decreasing sequence (3.12) is bounded from above. This means that there exists

$$S(x) = \lim_{n\to\infty} S_n(x) \quad \text{for } 0 \leqslant x \leqslant 1 - \varepsilon$$

We consider the question of the behavior of the function S in the neighborhood of the point $x = 1$. From (3.23) we find that the expression

$$\frac{\sigma(x)}{1 - \ln(1-x)}$$

should be bounded everywhere, that is

$$S_{n-1}(x) \leqslant \sigma(x) \leqslant A\ [1 - \ln(1 - x)] \tag{3.25}$$

Differentiating (3.1), substituting (3.25) into it, and letting $n \to \infty$, we find that $\Phi'(x)$ is bounded in the entire interval [0,1]; that is, there exists a lim $M_n = M$ for $n \to \infty$. Substituting (3.25) and (3.4) into (3.3) we obtain

$$S_n \leqslant \int_0^x \left\{ A^2 [1 - \ln(1-x)]^2 + \frac{M_n}{4k^2} \left[\frac{3 + 2\ln 2}{2} - \ln(1-x) \right] \right\} dx \tag{3.26}$$

Letting $n \to \infty$ we find that the right side of (3.26) is bounded as $x \to 1$. Hence it follows that the function $S(x)$ is continuous in the entire interval [0,1]. Thus it is proved that the successive approximations converge if

$$\frac{1}{k} < 4.8096 \tag{3.27}$$

Furthermore, it is not difficult to see that the limiting functions $S(x)$, $F(x)$ and $\Phi(x)$ satisfy the system of equations (1.25), (2.4) and the conditions (1.26). The solution obtained is unique. Indeed, the function F in (1.25) is represented analytically through S, because the right side of (1.25) satisfies a Lipschitz condition in the interval [0,1]. This guarantees the uniqueness of the solution.

It remains to show that the solution found satisfies (1.17) and also very strong conditions of boundedness on v_z and $v_r = 0$ at $r = 0$.

From the preceding, $S(1) = N$ is bounded, and in conformity with (1.24)

$$y \sim 2kN\,(1 - x^2), \qquad y' \sim -4kN \qquad \text{as } x \to 1$$

Hence it is evident that (1.17) is satisfied. Using (1.14) we obtain

$$\begin{aligned} u &= \frac{rv_r}{C_0} \sim 2kN \frac{r^2}{r^2+z^2} \\ w &= \frac{rv_z}{C_0} \sim 4kN \frac{r}{\sqrt{r^2+z^2}} \end{aligned} \qquad \begin{pmatrix} r \to 0 \\ z \neq 0 \end{pmatrix}$$

which also gives a useful result. Finally, using (1.11) and (1.12) it is easy to find that lim $\pi = -1/2$ as $\eta \to \infty$.

Thus is proved: for Reynolds numbers less than 4.8096 the problem posed has a solution that is unique and continuous everywhere except at the origin of coordinates; for Reynolds numbers greater than 8 a bounded solution of the problem does not exist.

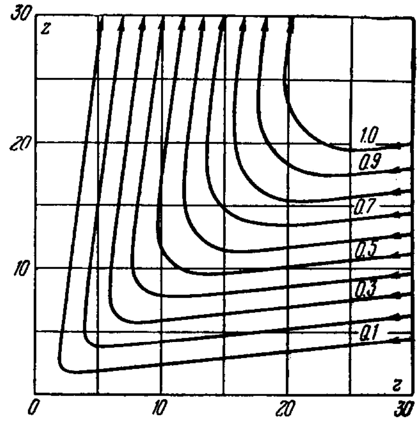

Fig. 3.

4. Approximate solution of the problem for very small Reynolds number. If $k \to \infty$, then $\Phi(x) \to x$ according to (1.23). Thus $\Phi(x) \approx x$ for very large k; then $F(x) = F_1(x)$ (2.8). According to (1.25) $S' = S^2$ for $k = \infty$; this together with the condition $S(0) = 0$ gives $S \neq 0$, and means also that $y \equiv 0$. Let $k \neq \infty$, but sufficiently large that the term y^2 in Equation (1.22) can be neglected. Then

$$2k\,(1 - x^2)\,y' + 4kxy = (4\ln 2 - 2)\,x + 2x^2 - (1 - x)^2 \ln(1 - x) - \\ - (1 + x)^2 \ln(1 + x) \qquad (4.1)$$

The solution of (4.1) with the condition $y(0) = 0$ has the form

$$y = \frac{1}{2k}\left[x + (2\ln 2 - 1)\,x^2 - \frac{1 + x^2}{2}\ln\frac{1 + x}{1 -} - x\ln(1 - x^2)\right] \qquad (4.2)$$

Thus $y \sim k^{-1}$, and in the left side of (1.22) we have neglected a term of $0(k^{-2})$ in comparison with terms of $0(1)$. We introduce into consideration the stream function in the meridional section

$$\Psi = \frac{k}{C_0}\int_0^r r v_z dr = k\int_0^r w\,dr = kz\int_\eta^\infty w\,\frac{d\eta}{\eta^2} = kr\,(w - \eta u) = k\sqrt{r^2 + z^2}\,y = kRy$$

$$\Psi = \frac{R}{2}\left[x + (2\ln 2 - 1)\,x^2 - \frac{1 + x^2}{2}\ln\frac{1 + x}{1 - x} - x\ln(1 - x^2)\right] \qquad (4.3)$$

Bearing in mind that $x = \cos\theta$, where θ is the angle measured in the meridional plane from the positive Z-axis, and using (4.3), it is easy to construct the streamlines, shown in Fig. 3 for equally-spaced values of Ψ: 0.1, 0.2, ... 1. As is evident, the character of the secondary flow corresponds to the ideas given in the introduction.

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